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The Algebra of Invariants of 3×3 Matrices over a Field of Arbitrary Characteristic

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Abstract

The least upper bound on degrees of elements of a minimal system of generators of the algebra of invariants of 3×3 matrices is found, and the nilpotency degree of a relatively free finitely generated algebra with the identity $x^3 = 0$ is established.

1 Introduction

Let K be an infinite field of arbitrary characteristic. Let a reductive algebraic group G act regularly on m-dimensional affine variety $V = K^m$. This action defines natural action of G on the coordinate algebra $K[V]$: $(g \cdot f)(v) = f(g^{-1} \cdot v)$, where $f \in K[V]$, $g \in G$, $v \in V$. Denote by $R = K[V]^G$ the algebra of invariants of $K[V]$ with respect to the action of G. By the Hilbert–Nagata Theorem, it is a finitely generated graded subalgebra. But Hilbert's proof for the case $\mathrm{char}(K) = 0$, as well as Nagata's proof for the case $\mathrm{char}(K) > 0$, are not constructive. The goal of the constructive theory of invariants is to find a minimal (i.e. irreducible) homogeneous system of generators (MSG) of $K[V]^G$ explicitly. It is an important problem, which arose as early as the theory of invariants itself. If one knows generators for each homogeneous component

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of the algebra of invariants, then, theoretically, the problem of finding MSG is equivalent to finding a constant D such that $K[V]^G$ is generated by invariants of degree not greater than D [8]. Popov gave a bound D for a connected semisimple group in characteristic zero case [8]. But Popov's bound is rather rough, so the problem of finding finer bounds is open.

Let $N_0 = \{0, 1, 2, \ldots\}$. If A is a N_0-graded algebra, denote by A^+ the subalgebra generated by elements of A of positive degree. It is easy to see that the set $\{r_i\} \subseteq R$ is a MSG iff $\{\overline{r_i}\}$ is a basis of $\overline{R} = R/(R^+)^2$. Call an element $r \in R$ *decomposable* if it belongs to the ideal $(R^+)^2$. So the least upper bound for the degrees of elements of MSG of the algebra of invariants is equal to the highest degree of indecomposable invariants.

Let $G = GL_n(K)$ act on the affine space $M_{n,d}(K) = M_n(K) \oplus \cdots \oplus M_n(K)$ (d times) by conjugations according to the following rule: $B \cdot (A_1, \ldots, A_d) \to (BA_1B^{-1}, \ldots, BA_dB^{-1})$, where $M_n(K)$ is the space of all $n \times n$ matrices over K, $A_i \in M_n(K)$, $B \in GL_n(K)$ ($i = \overline{1,d}$). This action induces an action on the coordinate ring $K_{n,d} = K[x_{ij}(r) \mid i, j = \overline{1,n};\ r = \overline{1,d}]$. Denote by $R_{n,d} = K_{n,d}^G$ the algebra of invariants. Let $X_r = (x_{ij}(r))_{1 \le i,j \le n}$ be the generic matrices of order n, and let $\sigma_k(X)$ be the coefficients of the characteristic polynomial of an $n \times n$ matrix X

$$\chi_n(X) = X^n - \sigma_1(X)X^{n-1} + \cdots + (-1)^n \sigma_n(X)E.$$

It is easy to see that $\sigma_k(X_{i_1} \cdots X_{i_s}) \in R_{n,d}$. Denote by $D(n, d, K)$ (by $D_{\sigma_k}(n, d, K)$, respectively) the highest degree of indecomposable invariants (of indecomposable invariants of the form $\sigma_k(X_{i_1} \cdots X_{i_s})$, respectively).

In the case $\mathrm{char}(K) = 0$, $R_{n,d}$ has been well investigated. Its generators and relations are described in [12], [9], [10]. In particular, it is known that $R_{n,d}$ is generated by its elements of degree $\le n^2$ [10]. Great progress in the study of $R_{n,d}$ in the case of positive $\mathrm{char}(K)$ was made due to Donkin and Zubkov. Donkin showed that $R_{n,d}$ is generated by all elements of the form $\sigma_k(X_{i_1} \cdots X_{i_s})$ [4], and Zubkov [15] extended Procesi-Razmyslov's Theorem on the relations to this case. Before formulating this theorem, we must fix some notation.

Let A be an associative algebra. Denote by $A\text{-}\mathrm{alg}\{b_1, \ldots, b_d\}$ the associative algebra generated over A by $1, b_1, \ldots, b_d$ which commute with A. If $b_1, \ldots, b_d$ are free over A, i.e. all identities of $C = A\text{-}\mathrm{alg}\{b_1, \ldots, b_d\}$ are consequences of identities of A, identities $b_i a = a b_i$ ($i = \overline{1,d}$, $a \in A$) and identities of associativity, we denote C by $A\langle x_1, \ldots, x_d\rangle$. If $C = A\text{-}\mathrm{alg}\{b_1, \ldots, b_d\}$ is a N_0-graded algebra with a unit over a N_0-graded subalgebra A, and elements $b_1, \ldots, b_d$ are homogeneous of positive degree, we denote by $C^{\#}$ the graded subalgebra $\sum_{1 \le i \le d} C b_i$. The ideal generated by

$a_1,\ldots,a_d$ is denoted by $\mathrm{id}\{a_1,\ldots,a_d\}$. We call an identity $h=0$ of A a *consequence* of identities $\{h_i=0\}$ if $h=\sum_{j=1}^s a_j h_{k_j} a'_j$, where $a_j, a'_j \in A$. The homogeneous component of degree r of a graded algebra A is denoted by $A(r)$.

The algebra of *concomitants* $C_{n,d}$ for $M_{n,d}$ is the algebra of all polynomial $GL_n(K)$–equivariant mappings of the space $M_{n,d}(K)$ to $M_n(K)$, where $GL_n(K)$ acts on $M_{n,d}(K)$ and $M_n(K)$ by conjugation. It is easy to see that $C_{n,d}$ is isomorphic to $R_{n,d}$- $\mathrm{alg}\{X_1,\ldots,X_d\}$, i.e. the subalgebra of $M_n(K_{n,d})$ generated by the generic matrices $X_1,\ldots,X_d$ over $R_{n,d}$ [14]. For $k>n$, consider the embedding $M_n\to M_k$ taking $A\in M_n$ to the matrix whose left upper $n\times n$ cell coincides with A and all the other entries are equal to zero. This mapping induces homomorphisms $R_{k,d}(r)\to R_{n,d}(r)$ and $C_{k,d}(r)\to C_{n,d}(r)$ (consult [14] for details). Taking projective limits, we obtain the *free algebra of invariants* of d matrices $R_d=\bigoplus_{r\geq 0}\mathrm{projlim}_n\, R_{n,d}(r)$ and the *free algebra of concomitants* $C_d=\bigoplus_{r\geq 0}\mathrm{projlim}_n\, C_{n,d}(r)$. Let S be the free semigroup generated by letters $\{a_1,a_2,\ldots\}$. Words $b=a_{i_1}\cdots a_{i_l}$ and $c=a_{j_1}\cdots a_{j_l}$ are called equivalent, if there exists a cyclic permutation $\pi\in S_l$ such that $i_s=j_{\pi(s)}$, $s=\overline{1,l}$. The *cycle* (in letters $a_1,a_2,\ldots$) is the equivalence class of some word. The cycle is *primitive*, if it is not equal to a power of a shorter cycle. It is known that R_d is isomorphic to the algebra of polynomials in 'symbolic' free generators $\sigma_k(a)$, where a is a primitive cycle in letters $x_1,\ldots,x_d$, $k>0$ [5]. The algebra C_d is isomorphic to the free associative algebra in 'formal matrix variables' $x_1,\ldots,x_d$ over R_d, i.e. $C_d\simeq R_d\langle x_1,\ldots,x_d\rangle$ [14]. If $n\geq r$, then $R_{n,d}(r)$ is naturally isomorphic to $R_d(r)$, and $C_{n,d}(r)$ is isomorphic to $C_d(r)$ (see [5]). Denote by $\deg(c)$ the degree of a word c, i.e. the number of letters appearing in c, and by $\mathrm{mdeg}(c)$ the multidegree of c, i.e. $\mathrm{mdeg}(c)=(\lambda_1,\lambda_2,\ldots)$, where λ_j is the number of times a_j appears in c. These notations are also used for cycles.

Let B be an arbitrary commutative algebra, $q_1,\ldots,q_s\in B$, and let $A_1,\ldots,A_s$ be $n\times n$ matrices over B. For $k=\overline{1,n}$ Amitsur's formula states [1]:

$$\sigma_k(\sum_{l=1}^s q_l A_l)=\sum(-1)^{k-(j_1+\cdots+j_t)}q^{j_1\,\mathrm{mdeg}(c_1)+\cdots+j_t\,\mathrm{mdeg}(c_t)}\sigma_{j_1}(c_1)\cdots\sigma_{j_t}(c_t), \qquad (1)$$

where $q^{(\lambda_1,\ldots,\lambda_s)}=q_1^{\lambda_1}\cdots q_s^{\lambda_s}$, and the sum ranges over all pairwise different primitive cycles $c_1,\ldots,c_t$ in letters $A_1,\ldots,A_s$ and numbers $j_1,\ldots,j_t$ with $\sum_{i=1}^t j_i\deg(c_i)=k$. By (1) one can express $\sigma_k(G)\in R_{n,d}$ in terms of elements of the form $\sigma_k(U)$, where $G\in C_{n,d}^{\#}$, U is a non-empty word in the generic matrices. Identification $R_n(r)$ with $R_{n,d}(r)$ for $n\geq r$ allows one to define $\sigma_k(g)\in R_d$ for $g\in C_d^{\#}$, $k\geq 1$, correctly.

The algebra R_d can be regarded as an associative-commutative K–algebra with a unit generated by 'symbolic' elements $\sigma_k(g)$, $g\in K\langle x_1,\ldots,x_d\rangle^{\#}$, $k>0$. The ideal

of relations of the algebra R_d is generated by (see [16]):

(A) $\forall k \geq 1, \forall g, h \in K\langle x_1, \ldots, x_d\rangle^{\#}, \sigma_k(gh) = \sigma_k(hg)$.

(B) Amitsur's formula.

(C) $\forall \alpha \in K, \forall k \geq 1, \forall g \in K\langle x_1, \ldots, x_d\rangle^{\#}, \sigma_k(\alpha g) = \alpha^k \sigma_k(g)$.

(D) $\forall t, k \geq 1, \forall g \in K\langle x_1, \ldots, x_d\rangle^{\#}, \sigma_k(g^t) = \sum\limits_{i_1,\ldots,i_{kt}} \beta^{(k,t)}_{i_1,\ldots,i_{kt}} \sigma_1(g)^{i_1} \cdots \sigma_{kt}(g)^{i_{kt}}$,

where coefficients $\beta^{(k,t)}_{i_1,\ldots,i_{kt}} \in Z$ are determined uniquely.

The kernels of natural projections $R_d \to R_{n,d}$, $C_d \to C_{n,d}$ we denote by $I_{n,d}$, $J_{n,d}$, respectively. Procesi–Razmyslov's Theorem asserts that the ideal $I_{n,d}$ is generated by 'symbolic' elements $\sigma_k(f)$, $k > n$, and the ideal $J_{n,d}$ — by elements $\chi_k(f) = f^k - \sigma_1(f)f^{k-1} + \cdots + (-1)^k\sigma_k(f)$, $k \geq n$, where $f \in C_d^{\#}$. In other words, the ideal of relations of $R_{n,d}$ is generated by (A)–(E), where

(E) $\forall k > n, \forall f \in C_d^{\#}, \sigma_k(f) = 0$.

In [2] it is proved that, in contrast to the case of $\mathrm{char}(K) = 0$, if $0 < \mathrm{char}(K) \leq n$ then the degree bound for the generators of $R_{n,d}$ tends to infinity when d tends to infinity. In [2] an explicit MSG for $R_{2,d}$ is given. In [3] some upper and lower bounds on $D(n, d, K)$ are pointed out.

In this paper we consider the case $n = 3$. We find the least upper bound on degrees of elements of MSG of $R_{3,d}$ (except for the case of $\mathrm{char}(K) = 3$, $d = 6k + 1$, $k > 0$, where we estimate the least upper bound with error not greater than 1).

Theorem 2 *The least upper bound $D = D(3, d, K)$ on degrees of elements of a minimal system of generators of the algebra of invariants $R_{3,d}$ for $d > 1$ is equal to:*

if $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > 3$*, then* $D = 6$,

$$\textit{if } \mathrm{char}(K) = 2\textit{, then } D = \begin{cases} d+2 & , \quad d \geq 4 \\ 6 & , \quad d = 2 \text{ or } 3 \end{cases}$$

$$\textit{if } \mathrm{char}(K) = 3\textit{, then } D = \begin{cases} 3d & , \quad d \text{ even} \\ 3d-1 & , \quad d \equiv 3 \text{ or } 5 \pmod 6 \\ 3d-1 \text{ or } 3d & , \quad d \equiv 1 \pmod 6. \end{cases}$$

If $d = 1$, then $D = 3$.

The *nilpotency degree* of a graded algebra $A = \bigoplus_{j \geq 0} A(j)$, where $A(0) = K$, is the least C for which $a_1 \cdots a_C = 0$ for all $a_i \in A^+$ $(i = \overline{1, C})$. The idea of the proof of Theorem 2 consists in reduction of the problem of decomposability of certain invariants to the problem of equality to zero of certain elements of the

algebra $N_{n,d} = K\langle x_1, \ldots, x_d\rangle/\mathrm{id}\{ f^n \mid f \in K\langle x_1, \ldots, x_d\rangle^{\#}\}$ and, in particular, to the question of finding $C(n, d, K)$ — the nilpotency degree of $N_{n,d}$ (see Lemmas 3, 5, 6). In the case of characteristic zero $n(n+1)/2 \leq C(n, d, K) \leq n^2$ (see [7], [11]), and Kuzmin conjectured that $C(n, d, K) = n(n+1)/2$. This conjecture has been proved to be true for $n \leq 4$ [13]. For prime characteristic, there exists an upper bound on $C(n, d, K)$ [6]: $C(n, d, K) < (1/6)n^6 d^n$.

In Section 2 of this paper we prove the following theorem.

Theorem 1 *The nilpotency degree $C = C(3, d, K)$ of $N_{3,d}$ $(d > 1)$ equals:*

if $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > 3$*, then* $C = 6$*,*

$$\textit{if } \mathrm{char}(K) = 2, \textit{ then } C = \begin{cases} d+3 & , \quad d \geq 3 \\ 6 & , \quad d = 2, \end{cases}$$

$$\textit{if } \mathrm{char}(K) = 3, \textit{ then } C = \begin{cases} 3d+1 & , \quad d \text{ is even} \\ 3d \text{ or } 3d+1 & , \quad d \text{ is odd.} \end{cases}$$

These theorems show one more difference between the cases of characteristic zero and prime: for $\mathrm{char}(K) = 0$, $D(3, d, K) = C(3, d, K)$ $(d > 1)$, while for $\mathrm{char}(K) = 2, 3$ $D(3, d, K) < C(3, d, K)$ ($d > 1$, and $d \neq 2, 3$ if $\mathrm{char}(K) = 2$).

2 Associative algebra with the identity $a^3 = 0$

2.1 General remarks

In this section we compute the nilpotency degree of a relatively free finitely generated algebra $N_{3,d} = K\langle x_1, \ldots, x_d\rangle/I$, where $I = \mathrm{id}\{f^3 \mid\ f \in K\langle x_1, \ldots, x_d\rangle^{\#}\}$, satisfying

$$T_1(a) = a^3 = 0,\ a \in K\langle x_1, \ldots, x_d\rangle^{\#}.$$

We call x_i letters, and monomials in x_i words. By x, y, z we denote any triple of pairwise distinct letters. Throughout this section, all considered elements of $N_{3,d}$ are meant to be non-empty words, and all words are meant to belong to $N_{3,d}$, if we do not explicitly write otherwise. Small Greek letters (possibly with index) denote elements of K. Denote by p ($p = 0, 2, 3, \ldots$) the characteristic of the field K.

Since the ideal I is homogeneous, $N_{3,d}$ possesses natural N_0- and N_0^d-gradings by degrees and multidegrees, respectively, for which we use the same notations as in Introduction. The degree of a word u in a letter x we denote by $\deg_x(u)$. The multidegree $(\alpha, \ldots, \alpha)$ (d times) will also be denoted by $\alpha^{(d)}$.

Partial and complete linearization of $a^3 = 0$ gives the identities

$$T_2(a,b) = a^2b + aba + ba^2 = 0.$$
$$T_3(a,b,c) = abc + acb + bac + bca + cab + cba = 0.$$

Denote by $\mathcal{S}$ the system

$$\left\{\begin{array}{ccc} g_1T_1(f)g_2 & = & 0 \\ g_1T_2(f_1,f_2)g_2 & = & 0 \\ g_1T_3(f_1,f_2,f_3)g_2 & = & 0, \end{array}\right. \tag{$\mathcal{S}$}$$

where f, f_i are non-empty words ($i = 1,2,3$), words g_1, g_2 can be empty, and equalities are meant to hold modulo ideal I. Let $\mathcal{S}_\Lambda$ be the subsystem of $\mathcal{S}$ which consists of equations of multidegree Λ. For each word u of multidegree Λ, introduce a variable x_u, and regard system $\mathcal{S}_\Lambda$ as a homogeneous system of linear equations in $\{x_u\}$ over K. Clearly, if $\mathrm{mdeg}(u) = \Lambda$, then $u = 0$ in $N_{3,d}$ iff $x_u = 0$ for each solution of $\mathcal{S}_\Lambda$. If $h = 0$ is an equation from $\mathcal{S}_\Lambda$, by $h|_{\{a_u\}}$ we denote the result of substitution $\{x_u = a_u\}$ in $h = 0$, where $a_u \in K$.

We call a word *canonical* with respect to x_i, if it has one of the following forms: w_1, $w_1x_iw_2$, $w_1x_i^2w_2$, $w_1x_i^2ux_iw_2$, where subwords w_1, u, w_2 do not contain x_i, and subwords w_1, w_2 can be empty. If a word is canonical with respect to each x_i, we call it *canonical*.

Statement 1 *An arbitrary word $w \in N_{3,d}$ is equal to a sum of canonical words which belong to the same homogeneous component as w.*

Proof. $T_2(x_i, w) = 0$ implies

$$x_iwx_i = -x_i^2w - wx_i^2. \tag{2}$$

Since $T_2(x, xw) = 0$, it follows that

$$xwx^2 = -x^2wx. \tag{3}$$

Applying to each letter (2) and then (3), we obtain the required. $\triangle$

Remark 1 *The presentation of a word from Statement 1 does not have to be unique.*

Corollary 1 *If a word $w \in N_{3,d}$ contains more than 4 occurrences of some letter, then $w = 0$. In particular, the length of a non-zero word does not exceed $3d$.*

Hereafter, to specify the subword to which the identity is applied, we sometimes put it in parentheses. Also, if we need to split a word into a product of subwords, we insert dots in it. (For example, see the deduction of (8) from (6).) Moreover, we will apply Statement 1 to all words without reference.

Let us obtain some identities. Applying (2), we get

$$(xy)^2 = (xyx)y = -x^2y^2 - (yx^2y) = y^2x^2. \tag{4}$$

Further, we apply (4) to all subwords equal to $xyxy$ without reference. Besides that, (2) implies $(xayx)y = -x^2ayy - a(yx^2y) = -x^2ay^2 + ay^2x^2 + ax^2y^2$, and $xa(yxy) = -(xay^2x) - (xax)y^2 = x^2ay^2 + ay^2x^2 + x^2ay^2 + ax^2y^2$. Hence

$$x^2ay^2 = 0, \text{ if } p \neq 3. \tag{5}$$

By separate linearization of (5) with respect to x and with respect to y, we obtain

$$x^2abc + x^2acb = 0, abcx^2 + bacx^2 = 0, \text{ if } p \neq 3. \tag{6}$$

Applying (2), we get $(xux)vx = -x^2uvx - ux^2vx$, $xu(xvx) = -xux^2v - xuvx^2 = x^2uxv + x^2uvx$. Hence

$$-2x^2uvx = x^2uxv + ux^2vx. \tag{7}$$

2.2 The case of $N_{3,2}$ in characteristic different from 3

Statement 2 *If $p \neq 3$, then $C(3, 2, K) = 6$.*

Proof. Applying (3) and (5) to x^2y^2xy, we obtain that $p \neq 3$ implies $x^2y^2xy = 0$. Statement 3 concludes the proof. △

Statement 3 $x_1^2x_2^2x_1 \neq 0$ *for each p.*

Proof. Let us find a solution for $\mathcal{S}_{(3,2)}$ for which $x_{x_1^2x_2^2x_1} \neq 0$. Let $x_{x_1^2x_2^2x_1} = 1$, $x_{x_1x_2^2x_1^2} = -1$, $x_{x_1^2x_2x_1x_2} = -1$, $x_{x_2x_1x_2x_1^2} = 1$, $x_{x_1x_2x_1^2x_2} = 1$, $x_{x_2x_1^2x_2x_1} = -1$, $x_{x_1x_2x_1x_2x_1} = 0$ and $x_u = 0$ for any other word u of multidegree $(3, 2)$. It is easy to see that this is indeed a solution for every equation from $\mathcal{S}_{(3,2)}$. △

2.3 The case of characteristic equal to 0 or greater than 3

Proposition 1 *If $p = 0$ or $p > 3$, $d > 1$, then $C(3, d, K) = 6$.*

Proof. Equality $x_1x_2 \cdots x_5 = 0$ implies $x^2y^2x = 0$, which is a contradiction to Statement 2.

Applying (6), we get $x^2 \cdot a \cdot b \cdot cd = -x^2 \cdot a \cdot cd \cdot b$ and $(x^2abc)d = -(x^2acb)d = -x^2 \cdot ac \cdot b \cdot d = x^2 \cdot ac \cdot d \cdot b$. Hence

$$x^2abcd = 0, abcdx^2 = 0. \tag{8}$$

It follows that $x^2y^2ab = 0, abx^2y^2 = 0$. Linearization of these identities with respect to x gives

$$abx^2cd + bax^2cd = 0, abx^2cd + abx^2dc = 0. \tag{9}$$

Further, (6) and (9) imply $T_3(x^2ab, c, d) = cx^2abd + dx^2abc = 0$ and $T_3(x^2, a, b)cd = ax^2bcd + bx^2acd = 0$. These two identities together with (6) imply that

$$a_1x^2a_2a_3a_4 = \operatorname{sgn}\sigma \cdot a_{\sigma(1)}x^2a_{\sigma(2)}a_{\sigma(3)}a_{\sigma(4)}, \; \sigma \in S_4. \tag{10}$$

Note that the same is true for $abcx^2d$. Further, $T_3(ax^2b, c, d) = cax^2bd + dax^2bc = 0$ (see (10)). This identity together with (9) imply

$$a_1a_2x^2a_3a_4 = \operatorname{sgn}\sigma \cdot a_{\sigma(1)}a_{\sigma(2)}x^2a_{\sigma(3)}a_{\sigma(4)}, \; \sigma \in S_4. \tag{11}$$

Let $A = ax^2bcd$, $B = abx^2cd$, $C = abcx^2d$. Then (8), (10), (11) imply $T_3(x^2, a, bc)d = A + B + 2C = 0$, $dT_3(x^2, a, bc) = -2A - B - C = 0$, $T_3(x^2a, bc, d) = -A + B - 2C = 0$. Hence $A = B = C = 0$. Linearization of (8) and $A = B = C = 0$ gives $a_1 \cdots a_6 = \operatorname{sgn}\sigma \cdot a_{\sigma(1)} \cdots a_{\sigma(6)}$, $\sigma \in S_6$. It follows that $T_3(ab, cd, ef) = 0$ implies $abcdef = 0$. $\triangle$

2.4 The case of characteristic 2

Proposition 2 *If $p = 2$, then* $C(3, d, K) = \begin{cases} d+3 & , \; d \geq 3 \\ 6 & , \; d = 2. \end{cases}$

Proof. If $d = 2$, see Statement 2.

A word of multidegree $\Lambda = (\lambda_1, \ldots, \lambda_d)$, where $\lambda_i > 1$ ($i = \overline{1,3}$), is equal to 0 by (5). Let us show that a word u of multidgree Λ, where $\lambda_1 = 3, \lambda_2 > 1, \lambda_3 > 0$, is equal to 0. Applying Statement 1, we represent u as a sum of words containing

subwords x^2y^2xz (see (5),(3)). But $x^2 \cdot y^2 \cdot x \cdot z = /\text{see } (6)/ = x^2y^2zx = /\text{see } (3)/ = xy^2zx^2 = /\text{see } (5)/ = 0$.

Let $d = 3$. We have $x^2y^2x \neq 0$ (by Statement 3), and all words of degree 6 are equal to 0. Hence, the nilpotency degree is equal to 6.

Let $d \geq 4$. The longest words which can be non-zero are words of multidegrees $(2, 2, 1, \dots, 1)$ and $\Theta = (3, 1, 1, \dots, 1)$. Below we prove the existence of a non-zero word of multidegree Θ. $\triangle$

Statement 4 *$x_1^2x_2 \cdots x_dx_1 \neq 0$, where $d \geq 2$.*

Proof. Let $V = x_1^2x_2 \cdots x_dx_1$, $d \geq 2$. First let us show that $V \neq 0$ when $d \geq 4$, which will imply that if $x_1^2x_2x_1 = 0$ or $x_1^2x_2x_3x_1 = 0$, then the substitution $x_2 \to x_2x_3x_4$ or $x_3 \to x_3x_4$, respectively, leads to required contradiction.

Let $d \geq 4$. There exists a solution for $\mathcal{S}_\Theta$ for which $V \neq 0$, namely, take $\{x_u = F(u) | \operatorname{mdeg}(u) = \Theta\}$, where $F(u)$ is equal to the number of all subwords x_1^2 in u. For example, if $\deg_{x_1}(u) = \deg_{x_1}(v) = 0$, then $F(ux_1^3v) = 0$. Let

$$F(v, w) = \begin{cases} 1 & , \text{ if } v = v'x_1, w = x_1w' \\ 0 & , \text{ otherwise} \end{cases} .$$

Here subwords v', u' can be empty. We have $F(v_1 \cdots v_l) = \sum_{i=1}^{l} F(v_i) + \sum_{i=1}^{l-1} F(v_i, v_{i+1})$. Hence $g_1T_3(f_1, f_2, f_3)g_2|_{\{F(u)\}} = 0$. Let $g_1T_2(x_1, f)g_2 = 0$ and $g_1T_1(f)g_2 = 0$ be equations from $\mathcal{S}_\Theta$. As one can see $g_1T_2(x_1, f)g_2|_{\{F(u)\}} = F(g_1) + F(f) + F(g_2) + F(g_1, f) + F(f, g_2)$. We have $\deg_{x_1}(g_1fg_2) = 1$, so $F(g_1) = F(f) = F(g_2) = F(g_1, f) = F(f, g_2) = 0$. Hence, $g_1T_2(x_1, f)g_2|_{\{F(u)\}} = 0$. Clearly, $g_1T_1(f)g_2|_{\{F(u)\}} = 0$. So $\{x_u = F(u)\}$ is a solution for $\mathcal{S}_\Theta$. $\triangle$

Remark 2 *One can show that there exist non-zero words of multidegree* $(2, 2, 1, \dots, 1)$ *(namely, $x_1^2x_2^2x_3 \cdots x_d \neq 0$, where $d \geq 2$).*

2.5 The case of characteristic 3

Proposition 3 *If $p = 3$, then* $C(3, d, K) = \begin{cases} 3d + 1 & , \ d \text{ is even} \\ 3d \text{ or } 3d + 1 & , \ d \text{ is odd.} \end{cases}$

The proof will follow from Statement 6 and Corollary 2.

Statement 5 *Let $\sum \alpha_iu_i = 0$ be a homogeneous identity of degree 1 or 2 in x_k, $k \in \overline{1, d}$, which contains some other letters. Then the result of substitution $x_k = 1$ in $\sum \alpha_iu_i = 0$ is an identity.*

Proof. Let M be the set of all identities from system $\mathcal{S}$ of degree 1 or 2 in x_k. The identity $\sum \alpha_i u_i = 0$ is a consequence of identities from M. Set M does not contain identities $g_1 T_1(f) g_2 = 0$, where $\deg_{x_k}(f) \neq 0$. Hence the result of substitution $x_k = 1$ in any identity from M is an identity. $\triangle$

Consider a word $x^2 ux$, where $\deg_x(u) = 0$. Replacing x with $x + y$, where $\deg_y(u) = 0$, and taking the homogeneous component of degree 1 in x and 2 in y, we get $y^2 ux + yxuy + xyuy$. Substitution $y = 1$ gives $ux - xu$. This reasoning shows that linear function $\Pi_x(v_1 x^2 uxv_2) = v_1 uxv_2 - v_1 xuv_2$, where v_1, v_2 are any words, is defined correctly on all homogeneous components of $N_{3,d}$ of degree 3 in x. Let $W_{xy} = x^2 y^2 xy$. We will shorten $W_{x_i x_j}$ to W_{ij}, and Π_{x_i} to Π_i. We have

$$\Pi_i \Pi_j (W_{ij}) = x_i x_j - x_j x_i. \tag{12}$$

The element $u = x_{\pi(1)} \cdots x_{\pi(t)} \in K\langle x_1, \ldots, x_t \rangle$, $\pi \in S_t$, is called even if permutation π is even, and odd otherwise. Define $\operatorname{sgn} u = 1$ for even u and $\operatorname{sgn} u = -1$ for odd u. Denote by $|u|$ the length of u.

Lemma 1 *If $|v_1|$ and $|v_2|$ are both odd or both even, then*

$$\operatorname{sgn} uv_1 u' v_2 u'' = (-1)^{|v_1| \cdot |v_2|} \operatorname{sgn} uv_2 u' v_1 u'',$$

where words u, u', u'' can be empty.

Proof. The statement follows from $\operatorname{sgn} uw_1 w_2 u'' = (-1)^{|w_1| \cdot |w_2|} \operatorname{sgn} uw_2 w_1 u''$, where words u, u'' can be empty. $\triangle$

Statement 6 *The word $w_{2k} = W_{12} W_{34} \cdots W_{2k-1,2k}$ is not equal to 0, if $k \geq 1$.*

Proof. Assume that, on the contrary, $w_{2k} = 0$. Then let

$$h_{2k} = \Pi_1 \Pi_2 \cdots \Pi_{2k-1} \Pi_{2k} (w_{2k}) = (x_1 x_2 - x_2 x_1) \cdots (x_{2k-1} x_{2k} - x_{2k} x_{2k-1}) = 0.$$

If a word $u \in K\langle x_1, \ldots, x_t \rangle$ of multidegree $1^{(2t)}$ is even, let $N_+(u) = 1$ and $N_-(u) = 0$; if it is odd, let $N_+(u) = 0$ and $N_-(u) = 1$. Let us show that for every equation $h = g_1 T_3(f_1, f_2, f_3) g_2 = 0$ from $\mathcal{S}_\Lambda$ (where $\Lambda = 1^{(2k)}$) it is true that

$$h|_{\{N_+(u)\}} = h|_{\{N_-(u)\}} = 0. \tag{13}$$

It is enough to consider equations with $g_1 = g_2 = 1$, because for words uv_1, uv_2 of multidegree Λ if $\operatorname{sgn} v_1 = \operatorname{sgn} v_2$, then $\operatorname{sgn} uv_1 = \operatorname{sgn} uv_2$. There are two possibilities:

1) Among f_1, f_2, f_3 there are two words of odd length, for example, f_2 and f_3. Then by Lemma 1, $\operatorname{sgn} f_1f_2f_3 = -\operatorname{sgn} f_1f_3f_2$, $\operatorname{sgn} f_2f_1f_3 = -\operatorname{sgn} f_3f_1f_2$, $\operatorname{sgn} f_2f_3f_1 = -\operatorname{sgn} f_3f_2f_1$. Hence (13) is true.

2) Among f_1, f_2, f_3 there are two words of even length, for example, f_2 and f_3. Then by Lemma 1, words $f_1f_2f_3$, $f_1f_3f_2$, $f_2f_1f_3$, $f_2f_3f_1$, $f_3f_1f_2$, $f_3f_2f_1$ are all even or all odd, hence (13) is true.

Prove by induction on t that $h_{2t}|_{\{N_+(u)\}} = (-1)^{t+1}$, $h_{2t}|_{\{N_-(u)\}} = (-1)^t$. For h_2 it is obvious. Since $h_{2t} = h_{2(t-1)}(x_{2t-1}x_{2t} - x_{2t}x_{2t-1})$, we have $h_{2t}|_{\{N_+(u)\}} = h_{2(t-1)}|_{\{N_+(u)\}} - h_{2(t-1)}|_{\{N_-(u)\}}$ and $h_{2t}|_{\{N_-(u)\}} = h_{2(t-1)}|_{\{N_-(u)\}} - h_{2(t-1)}|_{\{N_+(u)\}}$. By induction hypothesis, we get what is required.

We found a solution for $\mathcal{S}_\Lambda$ on which h_{2k} is not equal to zero, which is a contradiction. $\triangle$

Corollary 2 *The word $W_{12}W_{34}\cdots W_{2k-1,2k}x_{2k+1}^2$ is not equal to zero if $k > 0$.*

Proof. The proof follows from Statements 5 and 6. $\triangle$

In Section 3 we will need Statement 8, which is formulated below. We have $0 = (xy)^3 = (xyx)(yxy) = $ /see (2)/ $= (x^2y + yx^2)(y^2x + xy^2) = $ /see (3)/ $= -x^2y^2xy - y^2x^2yx$. Hence

$$W_{xy} = -W_{yx}. \tag{14}$$

We will shorten W_{xy} to W.

Statement 7 *Any word of degree 3 with respect to x and y is equal to $\sum \alpha_i u_i W w_i$, where subwords u_i, w_i do not contain x and y.*

Proof. By Statement 1, it is enough to consider canonical words. For words of multidegree $(3,3)$, Statement 7 follows from (14). Let us prove it for words of multidegree $(3,3,1)$.

$$\begin{aligned}
T_3(x^2y^2, a, xy) = 0 \quad &\Rightarrow \quad x^2y^2axy = -aW - Wa.\\
T_2(xy, ayx) = 0 \quad &\Rightarrow \quad x^2y^2ayx = aW - Wa.\\
T_3(x^2, ay^2x, y) = 0 \quad &\Rightarrow \quad x^2ay^2xy = aW.\\
T_3(y^2, yx^2a, x) = 0 \quad &\Rightarrow \quad x^2y^2xay = Wa.
\end{aligned}$$

Consider identities of multidegree $(3,3,1,1)$. Identity $T_3(x^2a, xy^2, by) = 0$ implies $x^2axy^2by = abW + Wab - baW - Wba$. The latter identity, together with $T_3(a, b, W) = 0$, implies

$$x^2axy^2by = -abW - Wab + aWb + bWa.$$

We apply (7) to subwords which are put into parentheses.

$$\begin{aligned}
&x^2ay^2bxy = (x^2 \cdot a \cdot y^2b \cdot x)y = abW - Wab + bWa.\\
&x^2ay^2xby = (x^2 \cdot a \cdot y^2 \cdot x)by = -abW - Wab - aWb + bWa.\\
&x^2y^2axby = x^2(y^2 \cdot ax \cdot b \cdot y) = -abW + Wab + bWa.\\
&x^2ay^2byx = (x^2 \cdot a \cdot y^2by \cdot x) = -Wab + bWa.\\
&x^2y^2aybx = (x^2 \cdot y^2ay \cdot b \cdot x) = abW - bWa.
\end{aligned}$$

Likewise we obtain identities of multidegree $(3, 3, 1, 1, 1)$:

$$\begin{aligned}
&x^2ay^2bxcy = (x^2 \cdot a \cdot y^2b \cdot x)cy = abcW + acWb + bcWa - aWbc - Wabc.\\
&x^2ay^2bycx = (x^2 \cdot ay^2 \cdot byc \cdot x) = abcW - abWc - acWb + aWbc + bWac - Wabc.\\
&T_3(x^2axby^2, c, y) = 0 \Rightarrow\\
&x^2axby^2cy = cabW - abWc + caWb - cbWa - aWcb + bWac - cWab + Wcab.
\end{aligned}$$

Modulo the identities which we obtained, each word is equivalent to an element of the required form. △

Corollary 3 *Let w have degree 3 in x and y. Then the result of substitution $\{x \to y,\ y \to x\}$ is $-w$.*

Proof. Let us denote the result of substitution by u. Statement 7 implies $w = \sum \alpha_i u_i W_{xy} w_i =$ /see (14)/ $= -\sum \alpha_i u_i W_{yx} w_i = -u$. △

By Corollary 3 we have for any word u

$$W_{ij}uW_{kl} = W_{kl}uW_{ij}. \tag{15}$$

Thus every permutation of subwords of the form W_{ij}, $i \neq j$, does not change a word. So with abuse of notation we denote all such words by one symbol W, i.e. uWv equals $uW_{ij}v$ for some $i \neq j$ such that letters x_i, x_j are not contained in u, v.

Corollary 4 *Every word of multidegree $3^{(2k)}$ equals αW^k.*

Proof. See Statement 7. △

Corollary 5 *Every word of multidegree $3^{(2k+1)}$ equals αx^2WxW^{k-1}, $k > 0$.*

Proof. Identity $T_2(W, x) = 0$ implies $W^2x + WxW + xW^2 = 0$. Multiplying the latter identity first from the left and then from the right by x^2, we get

$$x^2W^2x = -x^2WxW, \ \ x^2W^2x = -Wx^2Wx \tag{16}$$

(see (3)). Thus

$$Wx^2Wx = x^2WxW. \tag{17}$$

Identities $T_2(W, xW) = xW^3 + W^2xW + WxW^2 = 0$ and $T_2(W, Wx) = W^3x + W^2xW + WxW^2 = 0$ imply that

$$xW^3 = W^3x. \tag{18}$$

Let $r \in \{0, 1, 2\}$, $s \geq 0$. Identities (18), (16) imply $x^2W^{3s+r}x = x^2W^rxW^{3s} = r \cdot x^2WxW^{3s+r-1}$, since $p = 3$. Owing to (17), we have

$$W^ix^2W^lxW^j = l \cdot x^2WxW^{i+j+l-1}, \text{ where } i, j, k \geq 0. \tag{19}$$

This formula and Statement 7 conclude the proof. △

Statement 8 *If* $\mathrm{mdeg}(uv) = 3^{(d)}$, $d = 2k$ *or* $d = 6m + 1$ $(k, m > 0)$, *then* $uv = vu$.

Proof. Owing to Statement 1, we may assume that the words u, v are canonical. We will prove the Statement by 'decreasing' induction on s, where s is the number of subwords of the form W in words u and v.

Induction base. Let $d = 2k$. If $s = k$, then $u = W^l$, $v = W^{k-l}$ $(0 < l < k)$, and Statement 8 follows from (15).

Let $d = 6m + 1$. If $s = 3k$, then $uv = W^ix^2W^lxW^j$, where $i + j + l = 3m$, and both subwords u and v are products of elements of the set $\{x^2, x, W, \ldots, W\}$. Consider all possibilities:

1) $uv = W^ix^2W^{l_1} \cdot W^{l_2}xW^j$. Identities (19) and (3) imply $uv = (l_1 + l_2)x^2WxW^{3m-1}$, $vu = -(j + i)x^2WxW^{3m-1}$. Since $i + j + l_1 + l_2 = 3m$, words u and v commute.

2) $uv = W^ix^2W^lxW^{j_1} \cdot W^{j_2}$.

3) $uv = W^{i_1} \cdot W^{i_2}x^2W^lxW^j$.

The last two cases are similar to the first one.

Induction step. Assume that x and y are not contained in any of the subwords W of words u and v. Up to change of notations, all possibilities can be reduced, by means of (3), to the following:

1) u contains x^2, x, y^2, y; v does not contain letters x, y.

2) $u = ax^2by^2c$, $v = dxeyf$.

3) $u = ax^2by^2c$, $v = dyexf$.

4) $u = ax^2bxc$, $v = dy^2eyf$.

5) $u = ax^2bxcy^2d$, $v = eyf$.

6) $u = ax^2by^2cxd$, $v = eyf$.

7) $u = ay^2bx^2cxd$, $v = eyf$.

Here words $a, \dots, f$ can be empty. Consider these cases:

1) is obvious.

2) Identity $uv = vu$ is equivalent to $a(x^2by^2c \cdot dxey)f = d(xeyf \cdot ax^2by^2)c$. Applying Statement 7 to the subwords in parentheses, we can see that the previous identity is equivalent to $abcdeWf + abeWcdf + acdeWbf - abWcdef - aWbcdef = defabWc + debWfac + dfabWec - deWfabc - dWefabc$. By induction hypothesis, $-abcdeWf + abeWcdf + acdeWbf + abWcdef - aWbcdef - acdebWf - abWecdf + abcdWef = 0$. Changing notations $b \to a$, $cd \to b$, $e \to c$, we can see that the latter identity follows from $-abcW - bcaW + abWc + acWb + bcWa + aWbc - aWcb - Wabc = 0$. The last identity was verified by means of a computer programme, which was written by means of Borland C++ Builder (version 5.0). The programme is available upon request from the author.

The rest of the possibilities can be treated likewise. $\triangle$

3 Matrix invariant algebra

3.1 Auxiliary results

Similarly to the definition of R_d in terms of projective limits, let $\overline{R_d} = \bigoplus_{r\geq 0} \operatorname{projlim}_n \overline{R_{n,d}}(r)$, or, equivalently, $\overline{R_d} = R_d/(R_d^+)^2$. The algebras $R_{n,d}$, $\overline{R_{n,d}}$, R_d, $K\text{-}\operatorname{alg}\{X_1, \dots, X_d\} \subset C_{n,d}$ possess the natural N_0-grading by degrees and N_0^d-grading by multidegrees, for which we use the same notations as in Introduction. If elements r_1, r_2 of $R_{n,d}$ (of R_d, respectively) are equal modulo the ideal $(R_{n,d}^+)^2$ ($(R_d^+)^2$, respectively), we write $r_1 \equiv r_2$. Since the ideal $(R_{n,d}^+)^2$ is homogeneous with respect to the N_0^d-grading, one can see that for every equality of the form $r \equiv 0$, $r \in R_{n,d}$, and N_0^d-homogeneous component r' of r, $r' \equiv 0$ is also true. As in Section 2, monomials in the generic matrices $X_i \in C_{n,d}$ are called words, and X_i — letters. The same terminology is used for elements of C_d. By letters U, V, W, possibly, with indices, we denote non-empty words in the generic matrices, if we do not explicitly write otherwise.

Lemma 2

$$\frac{C_{n,d}}{\operatorname{id}\{R_{n,d}^+\}} \simeq N_{n,d}.$$

Proof. As we mentioned in Introduction,

$$C_{n,d} \simeq \frac{C_d}{J_{n,d}} \simeq \frac{R_d\langle x_1, \dots, x_d\rangle}{\operatorname{id}\{\chi_k(f) | k \geq n,\ f \in R_d\langle x_1, \dots, x_d\rangle^{\#}\}}.$$

Let $f \in C_d^{\#}$ and $f = f' + f''$, where $f' \in K\langle x_1, \ldots, x_d\rangle^{\#}$, $f'' \in R_d^+ C_d^{\#}$. Then $\chi_k(f) \in C_d$ is equal to f'^k modulo the ideal $\mathrm{id}\{R_d^+\} \triangleleft C_d$, because $\sigma_j(g) \in R_d^+$ for every $g \in C_d^{\#}$, $j > 0$. Thus the ideal $J_{n,d}$ is equal to $\mathrm{id}\{f^n | f \in K\langle x_1, \ldots, x_d\rangle^{\#}\} \triangleleft C_d$ modulo the ideal $\mathrm{id}\{R_d^+\}$. It is easy to see that the preimage of the ideal $\mathrm{id}\{R_{n,d}^+\} \triangleleft C_{n,d}$ in C_d is equal to $\mathrm{id}\{R_d^+\} + J_{n,d}$. By the Theorem on Homomorphism and the two preceding remarks, we have

$$\frac{C_{n,d}}{\mathrm{id}\{R_{n,d}^+\}} \simeq \frac{C_d}{J_{n,d}} \Big/ \frac{\mathrm{id}\{R_d^+\} + J_{n,d}}{J_{n,d}} \simeq \frac{C_d}{\mathrm{id}\{R_d^+\} + J_{n,d}} \simeq$$

$$\simeq \frac{C_d}{\mathrm{id}\{R_d^+\} + \mathrm{id}\{f^n | f \in K\langle x_1, \ldots, x_d\rangle^{\#}\}} \simeq N_{n,d}. \triangle$$

The image of $G \in C_{n,d}$ in $N_{n,d}$ we denote by $\overline{G}$. We denote any triple of pairwise distinct generic matrices by X, Y, Z, and their images we denote by x, y, z. We assume that $\sigma_k(f)$, where $f \in C_d^{\#}$, is an element of $R_{n,d}$, unless it is stated otherwise.

We will use the fact that $\mathrm{tr}(XY)$ is a nongenerate bilinear form, namely, if $\mathrm{tr}(GX_d) = 0$, where $G \in M_n(K_{n,d-1})$, then $G = 0$.

Lemma 3 *Suppose that $G \in K\text{-}\mathrm{alg}\{X_1, \ldots, X_d\}^{\#}$. Then*
1) If $\overline{G} = 0$ in $N_{n,d}$ then $\sigma_k(GX)$ is decomposable, where $k > 0$.
2) If G does not contain X and $\mathrm{tr}(GX)$ is decomposable, then $\overline{G} = 0$ in $N_{n,d}$.

Proof. 1) Owing to Lemma 2, the identity $\overline{G} = 0$ implies $G = \sum_i r_i U_i$, where $r_i \in R_{n,d}^+$, $\deg(U_i) \geq 0$. Thus $\sigma_k(GX) = \sigma_k(\sum_i r_i U_i X)$ is decomposable (see (1)).

2) Let $\mathrm{tr}(GX)$ be decomposable. Since $R_{n,d}$ is homogeneous, we have $\mathrm{tr}(GX) = \sum \mathrm{tr}(U_i X) r_i$, where words U_i may be empty, $\deg_X(U_i) = 0$ and $r \in R_{n,d}^+$. Hence $\mathrm{tr}((G - \sum r_i U_i)X) = 0$. Since the trace form is nongenerate, we have $G = \sum r_i U_i \in \mathrm{id}\{R_{n,d}^+\}$. Therefore $\overline{G} = 0$. $\triangle$

Further, we assume $n = 3$. Word U is called *canonical*, if $\overline{U} \in N_{3,d}$ is canonical.

Lemma 4 *For every non-empty word U there exist decompositions*

$$\mathrm{tr}(U) \equiv \sum \alpha_i \mathrm{tr}(W_i), \quad \sigma_2(U) \equiv \sum \beta_j \sigma_2(U_j) + \sum \gamma_l \mathrm{tr}(V_l),$$

where W_i, U_j, V_l are canonical words. Moreover, homogeneity of ideal $(R_{3,d}^+)^2$ implies that multidegrees of $\mathrm{tr}(U)$ and $\mathrm{tr}(W_i)$ are all equal, and also that multidegrees of $\sigma_2(U)$, $\sigma_2(U_j)$ and $\mathrm{tr}(V_l)$ are all equal.

Proof. First we will prove the Lemma for the trace. Owing to Lemma 3 and linearity of the trace, one can prove formulas analogous to (2) and (3), namely, for $i = \overline{1,d}$ we have $\mathrm{tr}(V_1X_iVX_iV_2) \equiv -\mathrm{tr}(V_1X_i^2VV_2) - \mathrm{tr}(V_1VX_i^2V_2)$,

$$\mathrm{tr}(V_1X_iVX_i^2V_2) \equiv -\mathrm{tr}(V_1X_i^2VX_iV_2). \qquad (20)$$

Here one of words V_1, V_2 can be empty. Hence the Lemma is proved for the trace.

The proof for $\sigma_2(U)$ is similar, except that instead of the trace linearity we apply consequence of Amitsur's formula for σ_2: $\sigma_2(V_1+V_2) \equiv \sigma_2(V_1) + \sigma_2(V_2) - \mathrm{tr}(V_1V_2)$, and then we apply the proved part of the Lemma to $\mathrm{tr}(V_1V_2)$. $\triangle$

Lemma 5 *Suppose that $G \in K\text{-}\mathrm{alg}\{X_1,\ldots,X_d\}$, G does not contain X. Then*

1)If $\mathrm{tr}(GX^2)$ is decomposable then $gx+xg=0$ in $N_{3,d}$, where $g=\overline{G}$.

2)In the case $p \neq 2$, the converse is also valid.

Proof. 1) Substituting $X+Y$ for X, where X, Y are not contained in G, and taking the homogeneous component of degree 1 in X and in Y, we get $\mathrm{tr}(GXY) + \mathrm{tr}(GYX) \equiv 0$. Hence $\mathrm{tr}((GX+XG)Y) \equiv 0$. Lemma 3 concludes the proof.

2) By Lemma 3, we have $\mathrm{tr}((GX+XG)X) \equiv 0$. Hence, $2\mathrm{tr}(GX^2) \equiv 0$. $\triangle$

Lemma 6 *Let $\deg_X(U) = \deg_X(V) = 0$ and $u=\overline{U}$, $v=\overline{V}$.*

1)If $\mathrm{tr}(X^2UXV) \equiv 0$ then $ux^2v - 2vx^2u - x^2uv - uvx^2 = 0$ in $N_{3,d}$.

2)In the case $p \neq 3$, the converse is also valid.

Proof. 1) Substituting $X+Y$ for X in $\mathrm{tr}(X^2UXV) \equiv 0$, and taking the homogeneous component of degree 2 in X and of degree 1 in Y, we get $\mathrm{tr}(VX^2UY) + \mathrm{tr}(UXVXY) + \mathrm{tr}(XUXVY) \equiv 0$. Here words U, V do not contain X, Y. By Lemma 3, we have $vx^2u + uxvx + xuxv = 0$ in $N_{3,d}$. This, together with identity (2), yields the required equality.

2) By Lemma 3, we have $\mathrm{tr}(UX^2VX) - 2\mathrm{tr}(VX^2UX) \equiv 0$. The identity (20) gives $3\mathrm{tr}(X^2UXV) \equiv 0$. $\triangle$

Applying Amitsur's formula to $\sigma_2(U+V)$ and letting $V=U$, obtain

$$2\sigma_2(U) = \mathrm{tr}^2(U) - \mathrm{tr}(U^2). \qquad (21)$$

Lemma 7 *a) $\sigma_2(X)$, $\det(X)$ are indecomposable. In particular, $D_{\det}(3,d,K) = 3$ and $D(3,1,K) = 3$.*

b) $\mathrm{tr}(X^2)$ is decomposable $\Leftrightarrow p = 2$.

c) $\mathrm{tr}(X^3)$ is decomposable $\Leftrightarrow p = 3$.

Proof. *a*) Let $\sigma_k(X)$ be decomposable ($k = 2, 3$). Then, by *D*), $\sigma_k(X)$ can be expressed in terms of $\sigma_l(X)$, $l < k$. Substitution $X = \mathrm{diag}(x_1, x_2, x_3)$ yields a contradiction to the fact that an elementary symmetric polynomial can not be expressed in terms of other elementary symmetric polynomials.

Lemma 3 implies $\mathrm{tr}(X^k) \equiv 0$, $k > 3$, and *D*) implies $\sigma_2(X^k) \equiv 0$, $k > 1$.

b) For $p = 2$, by (21), $\mathrm{tr}(X^2) \equiv 0$. For $p \neq 2$, if $\mathrm{tr}(X^2) \equiv 0$, then $2x = 0$ in $N_{3,d}$ (see Lemma 5), which is false.

c) If $\mathrm{tr}(\chi_3(X)) = 0$, then $\mathrm{tr}(X^3) \equiv 3\det(X)$. $\triangle$

3.2 The case of characteristic equal to 0 or greater than 3

Statement 9 *If $p = 0$ or $p > 3$, $d > 1$, then $D(3, d, K) = 6$.*

Proof. Element $\mathrm{tr}(X_1 \cdots X_7)$ is decomposable by Lemma 3 and identity $x_1 \cdots x_6 = 0$ in $N_{3,d}$. Hence $\mathrm{tr}(U) \equiv 0$ for every U with $\deg(U) > 6$.

Let us prove that $\mathrm{tr}(X^2Y^2XY)$ is indecomposable. Assume that it is decomposable. Letting $U = X^2$, $V = X$ and applying Lemma 6, we obtain $x^2y^2x - 2xy^2x^2 = 0$ in $N_{3,d}$. Then $x^2y^2x = 0$ (see (3)). But this yields a contradiction (see Statement 3).

Formula (21) concludes the proof. $\triangle$

3.3 The case of characteristic equal to 2

Proposition 4 *If $p = 2$, then* $D(3, d, K) = \begin{cases} d+2 & , \quad d \geq 4 \\ 6 & , \quad d = 2 \text{ or } 3 \end{cases}$.

The proof is a consequence of the following two statements.

Statement 10 *If $p = 2$, then* $D_{\mathrm{tr}}(3, d, K) = \begin{cases} d+2 & , \quad d \geq 4 \\ 6 & , \quad d = 2 \text{ or } 3 \end{cases}$.

Proof. By Lemma 4, it is sufficient to consider $\mathrm{tr}(U)$, where U is canonical.

First we point out some restrictions on the multidegree of an indecomposable invariant, namely, $\mathrm{tr}(U)$ is decomposable if multidegree of U is equal to

a) $\Delta_1 = (2, 2, 2, i_4, \ldots, i_d)$, where $i_4 + \cdots + i_d \geq 1$.

b) $\Delta_2 = (3, 2, i_3, i_4, \ldots, i_d)$, where $i_3 + \cdots + i_d \geq 2$.

Let us prove it. Every canonical word of multidegree Δ_1 is equal to $U_1X^2_{\pi(1)}U_2X^2_{\pi(2)}U_3X^2_{\pi(3)}U_4$, $\pi \in S_3$, where some words $U_1, \ldots, U_4$ (but not all of them) can be empty. Formula (5) and Lemma 3 yield $\mathrm{tr}(X^2V_1Z^2V_2) \equiv 0$. Hence, decomposability is established for *a*).

Let the multidegree of U be equal to $(3,2,1,1,i_5,\ldots,i_d)$. Then $\mathrm{tr}(U)$ is decomposable, because each word from $N_{3,d}$ of multidegree $(3,2,1,j_4,\ldots,j_d)$ is equal to 0 (see Section 2.4); then we apply Lemma 3, which gives decomposability for b).

Let $d=2$. Then $\mathrm{tr}(X^2Y^2XY)$ is a maximal indecomposable element (otherwise $y^2x^2y=0$ in $N_{3,d}$, by Lemma 6, but this is a contradiction, by Statement 3). Invariants of greater degree are, evidently, decomposable.

Let $d=3$. Then $\mathrm{tr}(X^2Y^2Z^2)$ is a maximal indecomposable element (otherwise Lemma 5 implies $x^2y^2z+zx^2y^2=0$ in $N_{3,d}$, thus $x^2y^2x=xx^2y^2=0$ in $N_{3,d}$, by (3), but $x^2y^2x\neq 0$ — see Statement 3). All invariants of greater degree are decomposable by b). Also note that $\mathrm{tr}(X^2Y^2XZ)$ is indecomposable, because, assuming that it is decomposable and letting $Z=Y$, we get that $\mathrm{tr}(X^2Y^2XY)$ is decomposable.

Let $d\geq 4$. Invariant $\mathrm{tr}(X_1^2X_2X_1X_3\cdots X_d)$ is a maximal indecomposable element, because $x_1^2x_2x_1x_3\cdots x_{d-1}\neq 0$ in $N_{3,d}$ (see Statement 4 and Lemma 3). All words of greater degree are decomposable by a) and b). Note that $\mathrm{tr}(X_1^2X_2X_1X_3\cdots X_d)$ is indecomposable (see Remark 2 and Lemma 3). $\triangle$

Statement 11 *If $p=2$, then* $D_{\sigma_2}(3,d,K)=\left\{\begin{array}{ccc} 6 & , & d\geq 3\\ 4 & , & d=2\end{array}\right.$.

Proof. Applying Amitsur's formula to $\sigma_4(u+v)=0$, where $u,v\in K\langle x_1,\ldots,x_d\rangle^{\#}$ are words, and considering the result modulo the ideal $(R^+_{3,d})^2$, we obtain $\sigma_2(UV)+\mathrm{tr}(U^3V)+\mathrm{tr}(V^3U)+\mathrm{tr}(U^2V^2)\equiv 0$, where U,V are non-empty words in the generic matrices. Since $\mathrm{tr}(U^3V)$ is decomposable (see Lemma 3), we have

$$\sigma_2(UV)\equiv \mathrm{tr}(U^2V^2). \tag{22}$$

Letting $U=X^2$, we obtain $\sigma_2(X^2V)\equiv 0$ (see Lemma 3).

Let $U=X_1$, $V=X_2\cdots X_d$. Then (22), together with the identity of $N_{3,d}$ $(x_2\cdots x_d)^2-x_d^2\cdots x_2^2=0$ (which is a consequence of (4)), to which we apply Lemma 3, yields $\sigma_2(X_1\cdots X_d)\equiv \mathrm{tr}(X_1^2(X_2\cdots X_d)^2)\equiv \mathrm{tr}(X_1^2X_d^2\cdots X_2^2)$. Element $\mathrm{tr}(X^2Y^2)$ is indecomposable (otherwise Lemma 5 implies $x^2y+yx^2=0$ in $N_{3,d}$, thus $x^2y^2x=-y^2x^2x=0$, but the last identity contradicts Statement 3). This reasoning and Statement 10 imply that $\sigma_2(X_1\cdots X_d)$ is decomposable iff $d\geq 4$. By Lemma 4, the statement 11 is proved. $\triangle$

3.4 The case of characteristic equal to 3

Proposition 5 *If $p=3$, then* $D(3,d,K)=\left\{\begin{array}{ccc} 3d & , & d \text{ is even}\\ 3d-1 & , & d\equiv 3,5 \ (\mathrm{mod}\, 6)\\ 3d-1 \text{ or } 3d & , & d\equiv 1\ (\mathrm{mod}\, 6).\end{array}\right.$

To prove the proposition, we need more detailed study of identities of $\overline{R_{3,d}}$.

Lemma 8 *If $\sum \alpha_i u_i = 0$ in $N_{3,d}$, where u_i are words, then $\sum \alpha_i \mathrm{tr}(U_i) \equiv 0$ in $\overline{R_{3,d}}$, where $\overline{U_i} = u_i$. Note that this identity is a consequence of*

$$\mathrm{tr}(dT_1(a)e) \equiv 0,\ \mathrm{tr}(dT_2(a,b)e) \equiv 0,\ \mathrm{tr}(dT_3(a,b,c)e) \equiv 0,$$

where words $a, b, c \in K\langle x_1, \ldots, x_d\rangle^{\#}$, and words $d, e \in K\langle x_1, \ldots, x_d\rangle$.

Proof. Denote by A, B, C, D, E words in the generic matrices, where D and E can be empty. Owing to Lemmas 7 and 3, $\mathrm{tr}(DA^3E)$ is decomposable. Linearization yields $\mathrm{tr}(DT_2(A,B)E) \equiv 0,\ \ \mathrm{tr}(DT_3(A,B,C)E) \equiv 0$.

Hence, if $g = 0$ is an identity from $\mathcal{S}$, then $\mathrm{tr}(G) = 0$ holds in $\overline{R_{3,d}}$, where $\overline{G} = g$ (see Section 2.1). Since all identities of $N_{3,d}$ are consequences of system $\mathcal{S}$, every identity of $N_{3,d}$ has a counterpart in $R_{3,d}$. $\triangle$

Lemma 9 *All identities of the algebra $\overline{R_{3,d}} = R_{3,d}/(R^+_{3,d})^2$ are consequences of*

(a) $\mathrm{tr}(dT_1(a)e) \equiv 0,\ \mathrm{tr}(dT_2(a,b)e) \equiv 0,\ \mathrm{tr}(dT_3(a,b,c)e) \equiv 0.$

(b) $\mathrm{tr}(ab) \equiv \mathrm{tr}(ba).$

(c) $\sigma_2(a) \equiv \mathrm{tr}(a^2).$

(d) $\det(ab) \equiv 0.$

(e) $\sigma_k(a) \equiv 0,\ k > 3.$

Here words $a, b, c \in K\langle x_1, \ldots, x_d\rangle^{\#}$, and words $d, e \in K\langle x_1, \ldots, x_d\rangle$.

Proof. For $I \in \{A, B, C, D, E\}$ let $(\overline{I})$ be the identity obtained by factorization of (I) modulo the ideal $(R^+_{3,d})^2$. Denote by (I_w), $(\overline{I_w})$, respectively, those identities of type (I), $(\overline{I})$, respectively, in which $g, h \in K\langle x_1, \ldots, x_d\rangle^{\#}$ are words. Throughout this proof we denote by letters u, v, possibly with indices, non-empty words from $K\langle x_1, \ldots, x_d\rangle$. The ideal of relations of R_d is generated by (A_w) and (D_w) (see [16]). Thus for the proof it is sufficient to show that $(\overline{A_w})$, $(\overline{D_w})$, $(\overline{E})$ can be deduced from (a)–(e). For $(\overline{A_w})$ it is obvious. Consider $(\overline{D_w})$: $\sigma_k(u^t) \equiv \alpha_{k,t}\sigma_{kt}(u)$, where $k \geq 1$, $t \geq 2$.

Let $k = 1$, $t = 2$. Since $\sigma_2(u)$ is indecomposable (Lemma 7), we have $\alpha_{1,2} = 1$, and $(\overline{D_w})$ follows from (c).

Let $k = 1$, $t = 3$. Since $\mathrm{tr}(u^3)$ is decomposable and $\det(u)$ is indecomposable (Lemma 7), we have $\alpha_{1,3} = 0$, and $(\overline{D_w})$ follows from (a).

If $k = 1$, $t \geq 4$, then $(\overline{D_w})$ follows from (a), (e).

If $k = 2$, then $(\overline{D_w})$ follows from (c) and (a), (e).

If $k = 3$, then $(\overline{D_w})$ follows from (d), (e).

If $k \geq 4$, then $(\overline{D_w})$ follows from (e).

For the proof of deducibility of $(\overline{E})$ we need some properties of identities of $\overline{R_{3,d}}$.

1) If an identity of $\overline{R_{3,d}}$ $t(x_1,\ldots,x_d)\equiv 0$ can be deduced from (a)–(e), then the identity $t(r_1u_1,\ldots,r_du_d)\equiv 0$, where $r_i\in R_{3,d}$, can be deduced from (a)–(e).

Let us prove 1). By homogeneity of (a)–(e) we may assume $t(x_1,\ldots,x_d)\equiv 0$ to be N_0^d-homogeneous. Then the identity $t(r_1u_1,\ldots,r_du_d)\equiv 0$ has the form $r\cdot t(u_1,\ldots,u_d)\equiv 0$, $r\in R_{3,d}$. Clearly the latter identity is a consequence of (a)–(e).

2) Let u_i be words such that $\deg_{x_1}(u_i)\in\{1,2\}$, and let

$$\sum \alpha_i \mathrm{tr}(u_i)\equiv 0 \tag{23}$$

be an identity of $\overline{R_{3,d}}$. Then (23) follows from (a), (b). In particular, if (23) is an identity of $\overline{R_{3,d}}$ and $\deg(u_i)\neq 3s$, then (23) follows from (a), (b).

Let us prove 2). Identity (23) can be assumed to be homogeneous. Let $\deg_{x_1}(u_i)=1$. Rewrite identity (23) in the form $\sum\alpha_i\mathrm{tr}(v_ix_1)\equiv 0$, where words v_i can be assumed to be non-empty. By Lemma 3, $\sum\alpha_iv_i=0$ in $N_{3,d}$. But then $\sum\alpha_iv_ix_1=0$ in $N_{3,d}$, and Lemma 8 concludes the proof.

Let $\deg_{x_1}(u_i)=2$. Identity (23) can be deduced from an identity $\sum\beta_i\mathrm{tr}(v_ix_1^2)\equiv 0$ by (a) (see Lemma 4), where words v_i can be assumed to be non-empty. By Lemma 5, we have $\sum\beta_i(x_1v_i+v_ix_1)=0$ in $N_{3,d}$. Substituting x_1^2 for x_1, applying Lemma 8 and using (b), we get the required.

3) Denote by (a_*), (b_*), (c_*), (d_*) identities of $\overline{R_{3,d}}$ of the type (a)–(d), respectively, in which $a,b,c\in K\langle x_1,\ldots,x_d\rangle^{\#}$ and $d,e\in K\langle x_1,\ldots,x_d\rangle$ (here a,b,c,d,e are not necessarily words). Thus (a_*)–(d_*) follow from (a)–(d).

Let us prove 3). Deducibility of (a_*), (b_*) from (a), (b) is obvious. Owing to property 1), we can assume that $a=\sum_{i=1}^s x_i$, $b=\sum_{i=s+1}^t x_i$ in (c_*), (d_*).

Consider (c_*): $\sigma_2(\sum x_i)\equiv \mathrm{tr}((\sum x_i)^2)$. Owing to (c), identity (c_*) follows from some identity of $\overline{R_{3,d}}$ of the type $\sum\beta_i\mathrm{tr}(v_i)\equiv 0$, where v_i are words of degree 2. The latter identity follows from (a), (b) by property 2).

Consider (d_*): $\det((\sum_{i=1}^s x_i)(\sum_{i=s+1}^t x_i))\equiv 0$. By (d), identity (d_*) follows from some identity of $\overline{R_{3,d}}$ of the type $\sum\gamma_l\mathrm{tr}(w_l)\equiv 0$, where w_l are products of words x_ix_j $(i=\overline{1,s},\ j=\overline{s+1,t})$ and $\deg(w_l)=6$. Taking homogeneous components, we may assume that this identity is homogeneous of multidegree Θ.

Let $\Theta\neq(3,3)$. Then all words w_l have degree 1 or 2 in some letter x_r. Applying property 2), we conclude the proof for this case.

Let $\Theta = (3,3)$. Then for each l for some r, q we have $w_l = (x_r x_q)^3$, and the identity follows from (a). Thus 3) is proved.

Now we can show that $(\overline{E})$: $\sigma_k(h) \equiv 0$, where $k \geq 4$, $h = \sum_{i=1}^m r_i u_i$, $r_i \in R_d$, follows from (a)–(e).

By property 1), we can assume $r_i = 1$, $u_i = x_i$. Our proof is by induction on $k \geq 4$, and for a fixed k — by induction on m.

Induction base. Let us show that (a)–(e) imply $\sigma_k(x_1+x_2) \equiv 0$. Owing to (c)–(e), this identity is a consequence of some identity of $\overline{R_{3,d}}$ of the form $\sum \alpha_i \mathrm{tr}(v_i) \equiv 0$.

If $k = 4, 5$, then $\deg(v_i) = 4$ or 5. The proof is concluded, by property 2).

If $k = 6$, then, by (d) and (e), the considered identity follows from $-\sigma_2(x_1^2 x_2) - \sigma_2(x_1 x_2^2) + \mathrm{tr}(x_1^2 x_2^2 x_1 x_2) + \mathrm{tr}(x_2^2 x_1^2 x_2 x_1) \equiv 0$. Identities (c) and (a) imply $\sigma_2(x_1^2 x_2) \equiv 0$, $\sigma_2(x_1 x_2^2) \equiv 0$. Identity $\mathrm{tr}(x_1^2 x_2^2 x_1 x_2) + \mathrm{tr}(x_2^2 x_1^2 x_2 x_1) \equiv 0$ follows from (a), by Lemma 8 applied to $x_1^2 x_2^2 x_1 x_2 + x_2^2 x_1^2 x_2 x_1 = 0$ in $N_{3,d}$ (see (14)).

If $k \geq 7$, then for every i we have $\deg_{x_1}(v_i) > 3$ or $\deg_{x_2}(v_i) > 3$, thus $v_i = 0$ in $N_{3,d}$. Hence $\mathrm{tr}(v_i) \equiv 0$ follows from (a) (see Lemma 8).

Induction step. Consider identity of $\overline{R_{3,d}}$ $\sigma_k(x_1 + x_2) = \sigma_k(x_1) + \sigma_k(x_2) + \sum_j \alpha_j \sigma_{k_j}(u_j) \equiv 0$, where $k_j < k$. Let $g = \sum_{i=2}^m x_i$. The induction hypothesis yields $\sigma_k(x_2|_{x_2 \to g}) \equiv 0$, $\sigma_{k_j}(u_j|_{x_2 \to g}) \equiv 0$ $(k_j > 3)$ follow from (a)–(e). Because $\sigma_k(x_1 + x_2) \equiv 0$ is a consequence of (a)–(e), we have $\sum_{k_j \leq 3} \alpha_j \sigma_{k_j}(u_j) \equiv 0$ follows from (a)–(d). Hence $\sum_{k_j \leq 3} \alpha_j \sigma_{k_j}(u_j|_{x_2 \to g}) \equiv 0$ follows from (a)–(d), by property 3). The lemma is proved. $\triangle$

Statement 12 *Let U_i be words of equal multidegree Θ. Then*

$$\sum \alpha_i \mathrm{tr}(U_i) \equiv 0 \tag{24}$$

is an identity of $\overline{R_{3,d}}$ if and only if system $\mathcal{S}_\Theta$ and identities $uv = vu$, where $u, v \in K\langle x_1, \ldots, x_d \rangle^{\#}$ are words and $\mathrm{mdeg}(uv) = \Theta$, *imply that $\sum \alpha_i \overline{U_i} = 0$.*

Proof. $\Leftarrow$ Apply Lemma 8.

$\Rightarrow$ By Lemma 9, identity (24) follows from (a)–(e). Identities (24), (a), (b) do not contain $\sigma_k(a)$, $k \geq 2$, while identities (c)–(e) contain them. Every 'symbolic' element $\sigma_k(a)$, where $k \geq 2$, occurs in exactly one identity from (c)–(e). Then the derivation of (24) from (a)–(e) can be transformed into a derivation of (24) from (a), (b). The identities (24), (a), (b) are homogeneous, thus for derivation of (24) we only need identities (a), (b) of multidegree Θ. $\triangle$

Now we can prove Proposition 5.

Proof. By Lemma 4 and formula (21), it is sufficient to consider invariants of the form $\mathrm{tr}(U)$, where word U is canonical. All words of the form $X_i^2X_j^2X_iX_j$, $i \neq j$, are denoted by the same symbol W, and let $w = \overline{W}$ (see Section 2.5 for details).

Let $d = 2k$, $k > 0$. In $N_{3,d}$ $w^k \neq 0$, and also $uv = vu$, where $\mathrm{mdeg}(uv) = 3^{(2k)}$ (Statement 8). Statement 12 yields the indecomposability of $\mathrm{tr}(W^k)$.

Let $d = 2k+1$, $k > 0$. Invariant $\mathrm{tr}(X^2W^k)$ is indecomposable, because otherwise Lemma 5 implies that $x_1w^k+w^kx_1 = 0$ in $N_{3,d}$. Substitution $x_1 = 1$ (see Statement 5) yields $w^k = 0$, which is a contradiction to Statement 6.

Let $d = 6m + r$, $r \in \{3, 5\}$, $m > 0$. Let us show that if $\mathrm{mdeg}(U) = 3^{(d)}$, then $\mathrm{tr}(U) \equiv 0$. For $u = \overline{U}$ we have $u = \alpha v_d$, where $v_d = x^2wxw^{k-1}$, $d = 2k + 1$ (Corollary 5). Hence $\mathrm{tr}(U) \equiv \alpha\mathrm{tr}(V_d)$, where $\overline{V_d} = v_d$ (see Lemma 8). If $r = 3$, then (18) implies $\mathrm{tr}(V_d) \equiv \mathrm{tr}(X^2WW^{3m}X) \equiv 0$. If $r = 5$, then it is easy to see that $\mathrm{tr}(V_d) = \mathrm{tr}(X^2WXW^{3m+1}) = \mathrm{tr}(XWW^{3m}X^2W) \equiv \mathrm{tr}(XWX^2W^{3m}W) \equiv$ /see (3)/ $\equiv -\mathrm{tr}(X^2WXW^{3m+1})$. Hence $\mathrm{tr}(V_d) \equiv 0$. $\triangle$

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